ON THE HOLONOMY OF KALUZA-KLEIN METRICS

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ABSTRACT. We investigate Kaluza-Klein metrics with a recurrent light-like vector field over a pseudo-Riemannian manifold (B, g).

1. INTRODUCTION

Principal S^1-bundles have been used by physicists to unify Einsteins general relativity theory with electrodynamics. This theory, by their main contributors known as Kaluza-Klein theory, has been later on generalized to bundles where the fiber is no longer an abelian Lie group(Yang-Mills theory). We refer to [Ba] for a general picture of the mathematical backgrounds of Gauge theory.

We focus here in particular on the construction of pseudo-Riemannian manifolds with the so called Kaluza-Klein-metric. Given a pseudo-Riemannian manifold (B, g), we consider a principal S^1-bundle P over B together with a connection form A on it. For the Kaluza-Klein-metric a fundamental vector corresponding to the S^1-action has length ± 1 whereas the horizontal spaces, corresponding to the connection form A, inherit the metric from the base manifold. Finally any horizontal vector is orthogonal to any vertical vector(*i.e.* which is tangent to the fiber). The well known Boothby-Wang fibration now gives examples of such principal S^1-bundles over a projective symplectic manifold and it is known that if the base B is a projective Kähler manifold, P will carry a natural Sasakian structure. We can fix transverse holonomy of the Sasakian manifold by fixing the holonomy of the base manifold. We refer to [Bl] and [BG2] for a detailed treatment of Sasakian geometry.

It is now known that Lorentzian connections on a $n+1$-dimensional manifold either admit full $\mathfrak{so}(1, n)$-holonomy or there is a holonomy-invariant direction, case in which we speak of special Lorentzian holonomy. By the work of L. Bérard Bergery, A. Ikemakhen ([BBI]) and T. Leistner ([L1]) indecomposable special holonomy has been classified. In section 4 we explore more generally (*i.e.* in the pseudo-Riemannian case) principal S^1-bundles with a Kaluza-Klein-metric admitting a holonomy-invariant direction or similarly a recurrent vector field. Under the condition that the recurrent vector field projects fiberwise on a fixed direction, it turns out that the recurrent vector field has to be parallel and projects onto a Killing field on the base manifold and that P has to admit a flat connection. Inversely we can construct examples of pseudo-Riemannian manifolds with a recurrent vector field by

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considering S^1-bundles with a flat connection over a base manifold with a Killing field and form a Kaluza-Klein-metric. In particular we can consider base manifolds to be K-contact or Sasakian.

Finally we can iterate the construction: A first Boothby-Wang construction over a projective symplectic manifold yields a K-contact manifold, and a second S^1-bundle-construction yields a manifold on which there is a Kaluza-Klein-metric with a parallel vector field. In case the starting manifold was projective Kähler the special Lorentzian holonomy (of type II) is completely known.

2. TRANSVERSE CONNECTIONS

Let (B, g) be a pseudo-Riemannian manifold with Levi-Civita covariant derivative ∇.

Let ξ be a vector field on (B, g) such that $g(\xi, \xi) = \sigma$ with $\sigma = \pm 1$.

Note $\mathcal{D} = \mathcal{D}_\xi$ the sub-bundle $\xi^\perp$. Note $p(X)$ the orthogonal projection of the vector field X on B onto $\mathcal{D}$. Note $\nabla^\mathcal{D}$ the covariant derivative on the bundle $\mathcal{D}$ defined by: $\nabla^\mathcal{D}_\xi Y = p([\xi, Y])$, $\nabla^\mathcal{D}_X Y = p(\nabla_X Y)$ for Y a section of $\mathcal{D}$ and X orthogonal to ξ.

It can be readily checked that $\nabla^\mathcal{D}$ is really a covariant derivative and that $\nabla^\mathcal{D}$ is torsion-free in the sense: $\nabla^\mathcal{D}_X Y - \nabla^\mathcal{D}_Y X = p([X, Y])$ for X, Y sections of $\mathcal{D}$. Furthermore $\nabla^\mathcal{D}$ preserves the bundle metric on $\mathcal{D}$ obtained by restriction of g.

Note $Hol^\mathcal{D}_o$ the holonomy group of the transverse connection in the point o and $\mathfrak{hol}^\mathcal{D}_o$ its Lie algebra.

3. KALUZA-KLEIN METRICS

3.1. **Definitions and basic results.** Given a pseudo-Riemannian manifold (B, g), let ∇ be its Levi-Civita covariant derivative. Let $\mathfrak{g}$ be the Lie algebra of the group $G = U(1)$. S^1 is the underlying manifold of the group G. $\mathfrak{g}$ identifies to $i\mathbb{R}$. We consider (P, π, B) a principal S^1-bundle over the manifold B. Let $A : TP \to \mathfrak{g}$ be a connection form on the bundle P, i.e. for $g \in G$ and X a vector field on P, $A(X \cdot g) = Ad(g^{-1})A(X)$ $(= A(X)$ in this case$)$, and for $a \in \mathfrak{g}$, $\xi \in P$, $A(\frac{d}{dt}_{t=0}(\xi \cdot \exp(ta))) = a$.

Let v be the vector field on P such that $T\pi(v) = 0$ and $A(v)$ is constant equal to i.

For X a vector field on B, we call *horizontal lift of X* and note X^* the unique vector field on P such that $A(X^*) = 0$ and $T\pi(X^*) = X$.

Recall that the curvature form Ω in $\Gamma((T^*B \wedge T^*B) \otimes \mathfrak{g})$ of the connection form A verifies $\Omega(X, Y) := -A([X^*, Y^*])$. As S^1 is abelian, Ω is closed. We have:

Proposition 1. $[X^*, v] = 0$,
  $[X^*, Y^*] = [X, Y]^* + i\Omega(X, Y)v$.

Proof. Classic results. □

Fix $\sigma = \pm 1$. Note $\tilde{g}$ the pseudo-Riemannian metric $\pi^* g + \sigma A \otimes A$ on P, called in the following *Kaluza-Klein-metric*. Let D be the Levi-Civita covariant derivative corresponding to $\tilde{g}$.

Note ϕ the section of $End_B(TB)$ such that $g(\phi X, Y) = \sigma \frac{i}{2} \Omega(X, Y)$.

Proposition 2. $\sigma \frac{i}{2} (\nabla \Omega)(X, Y, Z) = g((\nabla_X \phi) Y, Z)$

Proposition 3. $D_v v = 0$,
$\quad D_v X^* = D_{X^*} v = (\phi X)^*$
$\quad D_{X^*} Y^* = (\nabla_X Y)^* + \frac{i}{2} \Omega(X, Y) v.$

Proof. Use the Koszul formula. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 4. v is a Killing vector field s.t. $\tilde{g}(v, v) = -\sigma$.

Proposition 5. *Let $\mathcal{D}$ be the transverse bundle $v^\perp$. The transverse connection $D^{\mathcal{D}}$ verifies: $D^{\mathcal{D}}_v Y^* = 0$, $D^{\mathcal{D}}_{X^*} Y^* = (\nabla_X Y)^*$.*

The holonomy group in the point o of the connection $D^{\mathcal{D}}$ verifies $Hol^{\mathcal{D}}_o = \pi^*(Hol^{\nabla}_{\pi(o)})$.

3.2. Boothby-Wang fibration.

Consider the S^1-bundle P as in the preceding section. It is known (see chapter 2 in [Bl], resp. [K]) that $c_1(P) = [\frac{1}{2\pi i} \Omega]$ is an integral second De Rham cohomology class *i.e.* which lies in $H^2(B, \mathbb{Z})_b \subset H^2(B, \mathbb{R})$ (b stands for Betti-part).

Inversely suppose that Ω is a closed $i\mathbb{R}$-valued 2-form such that $[\frac{1}{2\pi i} \Omega] \in H^2(B, \mathbb{Z})_b$. As the principal S^1-bundles over B are classified by $H^2(B, \mathbb{Z})$, we can consider a bundle P corresponding to $[\frac{1}{2\pi i} \Omega]$. If B is simply connected, then the bundle P is uniquely defined. There is a (generally not unique) connection form A on the bundle P such that Ω is the curvature form of A.

3.3. Examples of Kähler base manifolds.

If (B, g, Ω) is a Kähler manifold, the condition $[\frac{1}{2\pi i} \Omega] \in H^2(B, \mathbb{Z})_b$ means, by the Kodaira embedding theorem, that B is projective, *i.e.* holomorphically embeds into some projective space $\mathbb{C}P^n$. The construction of section 3.1 gives then examples of Sasakian manifolds (see 4.5.2) with holonomy of the transverse bundle given by the holonomy of (B, g): By a theorem of Y. Hatakeyama, the total space of the principal S^1-bundle is normal contact if and only if the base manifold is Kähler and projective(see [H]).

By the Berger theorem it is known that the list of holonomy algebras of irreducible, non locally-symmetric Kähler manifolds is restricted to $\mathfrak{u}(n)$, $\mathfrak{su}(n)$ (Calabi-Yau), $\mathfrak{sp}(n)$ (hyperKähler).

3.3.1. *Projective Kähler symmetric spaces.*

Irreducible Kähler symmetric spaces are automatically Einstein. Furthermore if they are compact and simply-connected, they are projective(see [Be], 8.89, 8.2, 8.99). See also ([Be], 10.K) for the list of compact irreducible Kähler symmetric spaces.

3.3.2. *Projective base manifold with full $U(n)$-holonomy.*

Consider (X, g) a compact homogeneous Kähler manifold. If X is simply-connected, it is known (see [Be], 8.2, 8.98-99) that X admits a Kähler-Einstein metric g with positive scalar curvature, and X is projective. If (X, g) is not a symmetric space and irreducible, it admits necessarily then full $U(n)$-holonomy as Kähler-manifolds whose holonomy is included in $SU(n)$ are Ricci-flat.

3.3.3. *Projective Calabi-Yau base manifolds.* It is known that manifolds with full $SU(n)$-holonomy are automatically projective as soon as their complex dimension is bigger than 3.

3.3.4. *Projective hyperKähler base manifolds.* Very little examples are known by today of projective hyperKähler manifolds. One series of such can be obtained in the following way: Take X a projective K3-surface *e.g.* the Fermat quartic:

$$\{(x_0 : x_1 : x_2 : x_3)|x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\} \subset \mathbb{C}P^3$$

X itself and the Hilbert scheme $Hilb^n(X)$ of dimension $2n$ are then projective and hyperKähler. See ([GHJ], 21.1-2 and 26, and [Be] 14.C) for more information about these examples and projective hyperKähler manifolds in general.

4. Kaluza-Klein metrics with a recurrent vector field

4.1. **Recurrent vector fields.** We say that the non-vanishing vector field $\tilde{R}$ on P is *recurrent* if there is a scalar-valued 1-form ω on P such that $D\tilde{R} = \omega \otimes \tilde{R}$.

Note that if $\tilde{R}$ is a recurrent vector field, then $\langle \tilde{R} \rangle$ and $\tilde{R}^\perp$ are parallel distributions. In case the connection is torsion-free a parallel distribution is automatically integrable and tangent to a foliation of the manifold. In case $\tilde{R}$ is light-like, $\langle \tilde{R} \rangle$ is a sub-distribution of $\tilde{R}^\perp$ and the leaves corresponding to $\langle \tilde{R} \rangle$ are contained in the leaves corresponding to $\tilde{R}^\perp$.

4.2. **Screen bundle.** The quotient-bundle $\mathcal{S} := \tilde{R}^\perp / \langle \tilde{R} \rangle$ is called the *screen bundle* corresponding to the recurrent vector field $\tilde{R}$. Note q the corresponding canonical projection $\tilde{R}^\perp \to \mathcal{S}$. $\mathcal{S}$ carries a natural covariant derivative $D^\mathcal{S}$:

$$D_X^\mathcal{S}(qY) := q(D_X Y).$$

Lemma 6. $D^\mathcal{S}$ *is well defined.*

$D^\mathcal{S}$ restricts to any leaf $\mathcal{L}$ of the foliation $\mathcal{F}$ determined by $\tilde{R}^\perp$.

4.3. **Kaluza-Klein metrics with a recurrent vector field.**

> **Proposition 7.** *Given a pseudo-Riemannian manifold (B, g), and let σ be ± 1.*
>
> (i) *Let (P, π, B) a principal S^1-bundle over the manifold B and connection form A. Let $\tilde{g} := \pi^* g + \sigma A \otimes A$ be a metric on P.*
>
> *Suppose there is a light-like recurrent vector field $\tilde{R}$ on P of the form $f\xi^* + hv$, where ξ is a vector field on B such that $g(\xi, \xi) = \sigma$ and f and h are in $C^\infty(P)$.*
>
> *Then $\varepsilon = \mathrm{sign}(fh)$ is locally constant ± 1, and we have: ξ is a Killing vector field, $D(\xi^* + \varepsilon v) = 0$, $\phi\xi = 0$, $\Omega(\xi, \cdot) = 0$, $\phi = -\varepsilon \nabla \xi$,*
>
> *Furthermore $\Omega = -i\sigma\varepsilon d\eta$ for the 1-form $\eta := g(\xi, \cdot)$. The connection form $A_0 := A + i\sigma\varepsilon\pi^*\eta$ on the bundle $\pi : P \to B$ is flat.*

(ii) *Reciprocally suppose (B, g) admits a Killing vector field ξ such that $g(\xi, \xi) = \sigma$. Suppose the principal S^1-bundle $\pi : P \to B$ admits a flat connection A_0. Let $\tilde{g}$ be the metric $\pi^* g + \sigma A \otimes A$ for the connection form $A := A_0 - i\sigma\varepsilon\pi^*\eta$ with $\eta := g(\xi, \cdot)$ and $\varepsilon = \pm 1$. P admits then the parallel light-like vector field $\xi^* + \varepsilon v$ for the Levi-Civita connection associated to $\tilde{g}$.*

Notation: We will note r the parallel vector field $\xi^* + \varepsilon v$, and $\tilde{r}$ the complementary field $\xi^* - \varepsilon v$.

Proof. 1) The condition that $\tilde{R}$ is light-like writes $\tilde{g}(\tilde{R}, \tilde{R}) = 0$, giving $f^2 \cdot \sigma - h^2 \cdot \sigma = 0$, say $f = \varepsilon h$ with $\varepsilon = \pm 1$.

$$\tilde{R} = f \cdot (\xi^* + \varepsilon v)$$

As a consequence f is non vanishing.

2) $D\tilde{R} = \omega \otimes \tilde{R}$,

a) $D_v \tilde{R} = \omega(v)\tilde{R}$
$\Rightarrow \quad v(f)(\xi^* + \varepsilon v) + f \cdot (D_v \xi^* + \varepsilon D_v v) = \omega(v)f \cdot (\xi^* + \varepsilon v)$
$\Rightarrow \qquad v(f)(\xi^* + \varepsilon v) + f \cdot (\phi\xi) = \omega(v)f \cdot (\xi^* + \varepsilon v)$

From this follow the two equations:

$$v(f)v = \omega(v)fv \tag{1}$$
$$v(f)\xi^* + f \cdot (\phi\xi) = \omega(v)f\xi^* \tag{2}$$

Equation 1 gives

$$\omega(v) = v(\ln f) \tag{3}$$

From equation 2 follows

$$\phi\xi = 0 \tag{4}$$

otherwise stated

$$\Omega(\xi, \cdot) = 0 \tag{5}$$

b) $D_{X^*}\tilde{R} = \omega(X^*)\tilde{R}$

$$X^*(f)(\xi^* + \varepsilon v) + fD_{X^*}\xi^* + \varepsilon fD_{X^*}v = \omega(X^*)f\xi^* + \varepsilon\omega(X^*)fv$$

From this follow the two equations:

$$X^*(f)v = \omega(X^*)fv$$
$$X^*(f)\xi^* + f \cdot (\nabla_X \xi)^* + \varepsilon f \cdot (\phi X)^* = \omega(X^*)f\xi^*$$

then

$$X^*(f) = \omega(X^*)f \tag{6}$$
$$(\nabla_X \xi)^* = -\varepsilon(\phi X)^* \tag{7}$$

The two equations 3 and 6 give

$$\omega = d(\ln f) \tag{8}$$

From equation 8 follows in particular that the vector field $\tilde{R}_0 := \xi^* + \varepsilon v$ is parallel.

Equation 7 rewrites:

$$\phi X = -\varepsilon\nabla_X \xi \tag{9}$$

Using the definition of ϕ and the antisymmetry of Ω, one sees from this equation that ξ is a Killing vector field.

The relation $\Omega = -i\varepsilon\sigma d\eta$ follows from the fact: $d\eta(X,Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X,Y]) = 2g(-\nabla_X\xi, Y)$, true if ξ is a Killing vector field, and consequently $d\eta(X,Y) = 2\varepsilon g(\phi X, Y) = \sigma\varepsilon i\Omega(X,Y)$

The reverse implication is immediate after the preceding discussion. $\square$

Note that under the conditions of the proposition $r^\perp$ is exactly the horizontal distribution of the connection form A_0.

Proposition 8. *Under the conditions of proposition 7, we have:* $\mathcal{L}_\xi\eta = 0$, $\mathcal{L}_\xi d\eta = 0$, $\mathcal{L}_\xi\phi = 0$, $\nabla_\xi\phi = 0$.

Proof. Since $\mathcal{L}_\xi = d\circ\iota_\xi + \iota_\xi\circ d$, the two first statements follow from $\eta(\xi) = \sigma$ and $d\eta(\xi, \cdot) = 0$.

For the third note

$$0 = -\frac{\sigma\varepsilon}{2}(\mathcal{L}_\xi d\eta)(X,Y) = \xi g(\phi X, Y) - g(\phi[\xi, X], Y) - g(\phi X, [\xi, Y])$$

$$= (\mathcal{L}_\xi g)(\phi X, Y) + g((\mathcal{L}_\xi\phi)X, Y).$$

From this follows $\mathcal{L}_\xi\phi = 0$.

In particular it implies: $\nabla_\xi(\phi X) - \nabla_{\phi X}\xi = \phi(\nabla_\xi X - \nabla_X\xi)$ and so $(\nabla_\xi\phi)X = \nabla_{\phi X}\xi - \phi(\nabla_X\xi) = -\varepsilon\phi^2 X + \varepsilon\phi^2 X = 0$, proving the last statement. $\square$

Lemma 9. *Under the conditions of proposition 7, we have:* $(\mathcal{L}_{\phi X}\eta)(Y) = d\eta(\phi X, Y)$.

Proof. This follows from the definition of $\mathcal{L}$ and from $\eta(\phi X) = 0$. $\square$

4.4. **Transverse connection on $\xi^\perp$ in case D admits a recurrent vector field.** Note $\mathcal{D} = \mathcal{D}_\xi$ the sub-bundle $\xi^\perp$ and let $\nabla^{\mathcal{D}}$ be the transverse connection corresponding to ξ on the bundle $\mathcal{D}$.

Proposition 10. *Let D be as in proposition 7. Let X and Y be sections of $\mathcal{D}$.*

$$\begin{aligned}
Dr &= 0, \\
D_r Y^* &= (\nabla^{\mathcal{D}}_\xi Y)^*, \\
D_r \tilde{r} &= 0, \\
D_{X^*} Y^* &= (\nabla^{\mathcal{D}}_X Y)^* + \varepsilon\frac{i}{2}\Omega(X,Y)r, \\
D_{X^*}\tilde{r} &= -2\varepsilon(\phi X)^*, \\
D_v Y^* &= (\phi Y)^*, \\
D_v\tilde{r} &= 0.
\end{aligned}$$

Proposition 11. *Let D be as in proposition 7. Let X, Y, Z be sections of $\mathcal{D}$. The curvature of the covariant derivative D expresses by:*

$$
\begin{aligned}
R(\cdot, \cdot)r &= 0, \\
R(r, X^*)Y^* &= \left(R^{\mathcal{D}}(\xi, X)Y\right)^*, \\
R(r, X^*)\tilde{r} &= 0, \\
R(X^*, Y^*)Z^* &= \left(R^{\mathcal{D}}(X, Y)Z\right)^* + \sigma\varepsilon g\left((\nabla_X^{\mathcal{D}}\phi)Y - (\nabla_Y^{\mathcal{D}}\phi)X, Z\right)r, \\
R(X^*, Y^*)\tilde{r} &= -2\varepsilon\left((\nabla_X^{\mathcal{D}}\phi)Y - (\nabla_Y^{\mathcal{D}}\phi)X\right)^*, \\
R(r, v) &= 0, \\
R(X^*, v)Y^* &= \left((\nabla_X^{\mathcal{D}}\phi)Y\right)^* + \varepsilon\sigma g(\phi X, \phi Y)r, \\
R(X^*, v)\tilde{r} &= 2\varepsilon(\phi^2 X)^*.
\end{aligned}
$$

Let $\mathcal{S} := r^{\perp}/\langle r \rangle$ be the screen bundle corresponding to r. Note q the corresponding canonical projection.

As we saw before the covariant derivative $D^{\mathcal{S}}$ restricts to any leaf $\mathcal{L}$ of the foliation $\mathcal{F}$ associated to $\mathcal{D}$. Note $\mathfrak{hol}_o^{\mathcal{L},\mathcal{S}}$ its holonomy algebra.

Call ι the $C^\infty(B)$-linear mapping $\Gamma(TB) \to \Gamma(TP)$ defined by $\iota(\xi) = r$, $\iota(X) = X^*$, for X a section of $\mathcal{D}$. Note that ι is simply the horizontal lift associated to the connection form A_0. The mapping $l^{\mathcal{S}}$ (or simply l) shall be defined as follows: For $p \in P$ and $v \in \mathcal{D}_{\pi(p)}$, let $l_p(v) := q(\iota(X))_p$ for X a section of $\mathcal{D}$ such that $X_{\pi(p)} = v$. l_p is an isomorphism between $\mathcal{D}_{\pi(p)}$ and $\mathcal{S}_p$.

Proposition 12. *For X a vector field on B and Y a section of $\mathcal{D}$,* $D_{\iota X}^{\mathcal{S}} q(\iota Y) = q(\iota(\nabla_X^{\mathcal{D}} Y))$.
For $o \in \mathcal{L}$, $l_{o*}^{\mathcal{S}}(\mathfrak{hol}_{\pi(o)}^{\mathcal{D}}) = \mathfrak{hol}_o^{\mathcal{L},\mathcal{S}}$.

4.5. **Examples of base manifolds.**

4.5.1. *K-contact manifolds.* $(B^{2n+1}, \phi, \xi, \eta, g)$ is a K-contact manifold, meaning that:

(1) $(B^{2n+1}, \phi, \xi, \eta, g)$ is an almost contact metric manifold, *i.e.* ϕ, ξ, η are respectively a $(1,1)$-tensor, a vector field and a 1-form on the manifold B, g is a Riemannian metric on B, $\eta(\xi) = 1$, $d\eta(\xi, \cdot) = 0$, $\phi^2 = -I + \eta \otimes \xi$, $g(X, Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y)$,
(2) $g(\phi X, Y) = -d\eta(X, Y)$
(3) ξ is a Killing vector field

$(B^{2n+1}, \phi, \xi, \eta, g)$ verifies then the conditions of proposition 7(ii) with $\sigma = 1$ and $\varepsilon = 1$.

4.5.2. *Sasakian manifolds.* $(B^{2n+1}, \phi, \xi, \eta, g)$ is a Sasakian manifold, meaning that:

(1) $(B^{2n+1}, \phi, \xi, \eta, g)$ is a K-contact manifold
(2) ϕ satisfies $(\nabla_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X$

In this case we have $\nabla^{\mathcal{D}}\phi = 0$, and as a consequence $Hol_o^{0,\mathcal{D}} \subset U(\mathcal{D}_o)$. Additionally we see from proposition 11 that:

Proposition 13. *Suppose $(B^{2n+1}, \phi, \xi, \eta, g)$ is a Sasakian manifold. Let X, Y, Z be sections of $\mathcal{D} = \xi^{\perp}$. The curvature of the covariant derivative D on the manifold P constructed as before expresses by:*

$$
\begin{aligned}
R(\cdot, \cdot)r &= 0, \\
R(r, \cdot) &= 0, \\
R(X^*, Y^*)Z^* &= (R^{\mathcal{D}}(X, Y)Z)^*, \\
R(X^*, Y^*)\tilde{r} &= 0, \\
R(X^*, v)Y^* &= g(X, Y)r, \\
R(X^*, v)\tilde{r} &= -2X^*.
\end{aligned}
$$

Note that ϕ is parallel along any path γ. For X, Y sections of $\mathcal{D}$: $R(X^*, v)Y^* = g(X, Y)r$, as a consequence we obtain:

Proposition 14. *If $(B^{2n+1}, \phi, \xi, \eta, g)$ is a Sasakian manifold the holonomy algebra of the Levi-Civita covariant derivative D of g on the manifold P is of the form: For the decomposition $T_o P = \langle r_o \rangle \oplus (\iota \mathcal{D})_o \oplus \langle \tilde{r}_o \rangle$.*

$$
\mathfrak{hol}_o^D = \left\{ \begin{pmatrix} 0 & u^* & 0 \\ 0 & A & -u \\ 0 & 0 & 0 \end{pmatrix} \mid A \in \mathfrak{g}, u \in \mathrm{Hom}(\langle \tilde{r}_o \rangle, (\iota \mathcal{D})_o) \right\},
$$

where $\mathfrak{g} = \iota_{o*} \mathfrak{hol}_{\pi(o)}^{\mathcal{D}}$ is a sub-Lie-algebra of $\mathfrak{u}((\iota \mathcal{D})_o)$.

Note that in the theorem appears $\mathfrak{hol}_{\pi(o)}^{\mathcal{D}}$ and not $\mathfrak{hol}_{\pi(o)}^{\mathcal{L}, \mathcal{D}}$ like in proposition 12. In the proof, the fact that ϕ commutes with parallel transport on the transverse bundle is crucial.

5. Double bundles

The base manifold can be obtained by a preliminary circle bundle construction. We discuss this possibility in the following: Let (B, g) be a pseudo-Riemannian manifold with Levi-Civita connection ∇. Let $\pi_1 : P_1 \to B$ be a first principal S^1-bundle over B with connection form A_1 and metric $\tilde{g}_1 = \pi_1^* g + \sigma A_1 \otimes A_1$.

We have seen that the corresponding fundamental field $\xi = v$ is then a Killing field such that $\tilde{g}_1(v, v) = -\sigma$.

Let $\mathcal{D}_1$ be the transverse bundle $\xi^{\perp}$ equipped with the transverse covariant derivative D^1. Let ι_p be the mapping $T_{\pi_1(p)} B \to \mathcal{D}_{1,p}$ defined by $\iota_p(X_p) = X_p^*$ for X a vector field on B. ι_p is an isomorphism.

Consider a second principal S^1-bundle $\pi_2 : P_2 \to P_1$ admitting a flat connection form $A_{2,0}$. Fix $\varepsilon_2 := \pm 1$. Let $\eta_2 := \tilde{g}_1(\xi, \cdot)$ be a 1-form on P_1. Define the connection form A_2 by $A_2 := A_{2,0} + i\sigma \varepsilon_2 \pi_2^* \eta_2$ Let $\tilde{g}_2$ be the metric $\pi_2^* \tilde{g}_1 - \sigma A_2 \otimes A_2$. By proposition 7 follows that P_2 admits the parallel vector field $r = \xi^* + \varepsilon_2 v_2$ for the Levi-Civita connection associated to $\tilde{g}_2$.

Let $\mathcal{L}$ be a leaf of the foliation determined by $\mathcal{D}_2 := r^{\perp}$ and o be a point contained in it. Let $\mathcal{S}$ be the screen bundle determined by r.

Proposition 15. $\mathfrak{hol}_o^{\mathcal{L}, \mathcal{S}}$ *is isomorphic to* $\mathfrak{hol}_{\pi_1 \pi_2(o)}^{\nabla}$

Proof. By propositions 5 and 12 the holonomy algebra $\mathfrak{hol}_o^{\mathcal{L}, \mathcal{S}}$ verifies:
$\mathfrak{hol}_o^{\mathcal{L}, \mathcal{S}} = l_{o*}^{\mathcal{S}} (\iota_{\pi_2(o)*} (\mathfrak{hol}_{\pi_1 \pi_2(o)}^{\nabla}))$ $\square$

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